

A log-Birnbaum–Saunders Regression Model with Asymmetric Errors

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Abstract

The paper by Leiva et al. (2010) introduced a skewed version of the sinh-normal distribution, discussed some of its properties and characterized an extension of the Birnbaum–Saunders distribution associated with this distribution. In this paper, we introduce a skewed log-Birnbaum–Saunders regression model based on the skewed sinh-normal distribution. Some influence methods, such as the local influence and generalized leverage are presented. Additionally, we derived the normal curvatures of local influence under some perturbation schemes. An empirical application to a real data set is presented in order to illustrate the usefulness of the proposed model.

Key words: Birnbaum–Saunders distribution; fatigue life distribution; influence diagnostic; maximum likelihood estimators; sinh-normal distribution; skew-normal distribution.

1 Introduction

The two-parameter Birnbaum-Saunders (BS) distribution, also known as the fatigue life distribution, was introduced by Birnbaum and Saunders (1969a,b). It was originally derived from a model for a physical fatigue process where dominant crack growth causes failure. A more general derivation was provided by Desmond (1985) based on a biological model and relaxing several of the assumptions made by Birnbaum and Saunders (1969a). Desmond (1986) investigated the relationship between the BS distribution and the inverse Gaussian distribution. The author established that the BS distribution can be written as a mixture equally weighted from an inverse Gaussian distribution and its complementary reciprocal.

The random variable T is said to have a BS distribution with parameters $\alpha, \eta > 0$, say $\text{BS}(\alpha, \eta)$, if its cumulative distribution function (cdf) is given by $F(t) = \Phi(v)$, $t > 0$, where $\Phi(\cdot)$ is the standard normal distribution function, $v = \rho(t/\eta)/\alpha$, $\rho(z) = z^{1/2} - z^{-1/2}$ and α and η are shape and scale parameters, respectively. Also, η is the median of the distribution: $F(\eta) = \Phi(0) = 1/2$. For any constant $k > 0$, it follows that $kT \sim \text{BS}(\alpha, k\eta)$. It is noteworthy that the reciprocal property holds for the BS distribution: $T^{-1} \sim \text{BS}(\alpha, \eta^{-1})$; see Saunders (1974). The BS distribution has received

considerable attention over the last few years. Kundu et al. (2008) discussed the shape of the hazard function of the BS distribution. Results on improved statistical inference for the BS distribution are discussed in Wu and Wong (2004) and Lemonte et al. (2007, 2008). Some generalizations and extensions of the BS distribution are presented in Díaz–García and Leiva (2005), Gómez et al. (2009), Guiraud et al. (2009) and Castillo et al. (2009). This distribution has been applied in reliability studies (see, for example, Balakrishnan et al., 2007) and outside this field; see Leiva et al. (2008) and Leiva et al. (2009). Additionally, based on the BS distribution, Bhatti (2010) introduced the BS autoregressive conditional duration model. Xu and Tang (2010) presented estimators for the unknown parameters of the BS distribution using reference prior.

From Rieck (1989), if

$$Z = \nu + \frac{2}{\alpha} \sinh\left(\frac{Y - \gamma}{\sigma}\right) \sim N(0, 1), \quad (1)$$

then Y has a four-parameter sinh-normal (SHN) distribution, denoted by $Y \sim \text{SHN}(\alpha, \gamma, \sigma, \nu)$, where $\nu \in \Re$ and $\alpha > 0$ are the shape parameters, and $\gamma \in \Re$ and $\sigma > 0$ correspond to the location and scale parameters, respectively. According to Rieck (1989), the parameter ν is also the noncentrality parameter. If $\nu = 0$, the notation is reduced simply by $Y \sim \text{SHN}(\alpha, \gamma, \sigma)$, and this distribution has a number of interesting properties. For example, it is symmetric around the mean $E(Y) = \gamma$, it is unimodal for $\alpha \leq 2$ and bimodal for $\alpha > 2$ and if $Y_\alpha \sim \text{SHN}(\alpha, \gamma, \sigma)$, then $Z_\alpha = 2(Y_\alpha - \gamma)/(\alpha\sigma)$ converges in distribution to the standard normal distribution when $\alpha \rightarrow 0$. If $Y \sim \text{SHN}(\alpha, \gamma, \sigma = 2)$, then $T = \exp(Y)$ follows the BS distribution with shape parameter α and scale parameter $\eta = \exp(\gamma)$, i.e. $T = \exp(Y) \sim \text{BS}(\alpha, \eta)$. For this reason, according to Leiva et al. (2010), the SHN distribution is also called the log-Birnbaum–Saunders (log-BS) distribution. Additionally, according to these authors, the SHN and BS models corresponding to a logarithmic distribution and its associated distribution, respectively (Marshall and Olkin, 2007, Ch. 12).

Rieck and Nedelman (1991) introduced a log-BS regression model based on the $\text{SHN}(\alpha, \gamma, 2)$ distribution. Their regression model has been studied by several authors. Some important references are Tisonas (2001), Galea et al. (2004), Leiva et al. (2007), Desmond et al. (2008), Lemonte et al. (2010), Xiao et al. (2010) and Cancho et al. (2010), among others. Generalizations of the log-BS regression model introduced by Rieck and Nedelman (1991) are presented in Xi and Wei (2007, § 4) and Lemonte and Cordeiro (2009).

Leiva et al. (2010) introduced a skewed SHN distribution by replacing the standard normal distribution in equation (1) by the skew-normal (SN) distribution (Azzaline, 1985), i.e. they consider the random variable

$$Z = \nu + \frac{2}{\alpha} \sinh\left(\frac{Y - \gamma}{\sigma}\right) \sim \text{SN}(\lambda),$$

where $\lambda \in \Re$ is the shape parameter which determines the skewness. Now, the notation used is $Y \sim \text{SSN}(\alpha, \gamma, \sigma, \nu, \lambda)$. From now on, we shall consider $\nu = 0$ and $\sigma = 2$ and hence the notation is given by $Y \sim \text{SSN}(\alpha, \gamma, 2, \lambda)$. The random variable $T = \exp(Y)$ follows the extended Birnbaum–Saunders (EBS) distribution, with shape parameters $\alpha > 0$ and $\lambda \in \Re$, and scale parameter $\eta = \exp(\gamma)$. Now,

the notation is $T = \exp(Y) \sim \text{EBS}(\alpha, \eta, \lambda)$.

Let $T \sim \text{EBS}(\alpha, \eta, \lambda)$. The density function of $Y = \log(T)$ is given by (Leiva et al., 2010)

$$\pi(y) = \frac{2}{\alpha} \cosh\left(\frac{y - \gamma}{2}\right) \phi\left(\frac{2}{\alpha} \sinh\left(\frac{y - \gamma}{2}\right)\right) \Phi\left(\frac{2\lambda}{\alpha} \sinh\left(\frac{y - \gamma}{2}\right)\right), \quad y \in \mathbb{R},$$

where $\phi(\cdot)$ is the standard normal density function, and, as before, we write $Y \sim \text{SSN}(\alpha, \gamma, 2, \lambda)$. The s th ($s = 1, 2, \dots$) moment of Y can be written as

$$\text{E}(Y^s) = 2^k \sum_{k=0}^s \gamma^{s-k} c_k(\alpha, \lambda), \quad c_k(\alpha, \lambda) = \int_{-\infty}^{\infty} \{\sinh^{-1}(\alpha w/2)\}^k \phi(w) \Phi(\lambda w) dw.$$

Thus, the mean of Y is given by $\text{E}(Y) = \gamma + c(\alpha, \lambda)$, with

$$c(\alpha, \lambda) = 4 \int_{-\infty}^{\infty} \{\sinh^{-1}(\alpha w/2)\} \phi(w) \Phi(\lambda w) dw.$$

Plots of the $\text{SSN}(\alpha, \gamma, 2, \lambda)$ distribution are illustrated in Figure 1 for selected parameter values.

The chief goal of this paper is to introduce a skewed log-BS regression model based on the $\text{SSN}(\alpha, \gamma, 2, \lambda)$ distribution, recently proposed by Leiva et al. (2010). The proposed regression model is convenient for modeling asymmetric data, and it is an alternative to the log-BS regression model introduced by Rieck and Nedelman (1991) when the data present skewness. The article is organized as follows. Section 2 introduces the class of skewed log-BS regression models. The score functions and observed information matrix are given. Section 3 deals with some basic calculations related with local influence. Derivations of the normal curvature under different perturbation schemes are presented in Section 4. Generalized leverage is derived in Section 5. Section 6 contains an application to a real data set of the proposed regression model. Finally, concluding remarks are offered in Section 7.

2 Model specification

The skewed log-BS regression model is defined by

$$y_i = \mathbf{x}_i^\top \boldsymbol{\beta} + \varepsilon_i, \quad i = 1, \dots, n, \quad (2)$$

where y_i is the logarithm of the i th observed lifetime, $\mathbf{x}_i = (x_{i1}, \dots, x_{ip})^\top$ is a vector of known explanatory variables associated with the i th observable response y_i , $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)^\top$ is a vector of unknown parameters, and the random errors $\varepsilon_i \sim \text{SSN}(\alpha, -c(\alpha, \lambda), 2, \lambda)$ that corresponds to the regression model where the error distribution has mean zero. Thus, we have $y_i \sim \text{SSN}(\alpha, \mathbf{x}_i^\top \boldsymbol{\beta} - c(\alpha, \lambda), 2, \lambda)$, with $\text{E}(y_i) = \mathbf{x}_i^\top \boldsymbol{\beta}$, for $i = 1, \dots, n$.

The log-likelihood function for the vector parameter $\boldsymbol{\theta} = (\boldsymbol{\beta}^\top, \alpha, \lambda)^\top$ from a random sample $\mathbf{y} = (y_1, \dots, y_n)^\top$ obtained from (2) can be expressed as

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^n \ell_i(\boldsymbol{\theta}), \quad (3)$$

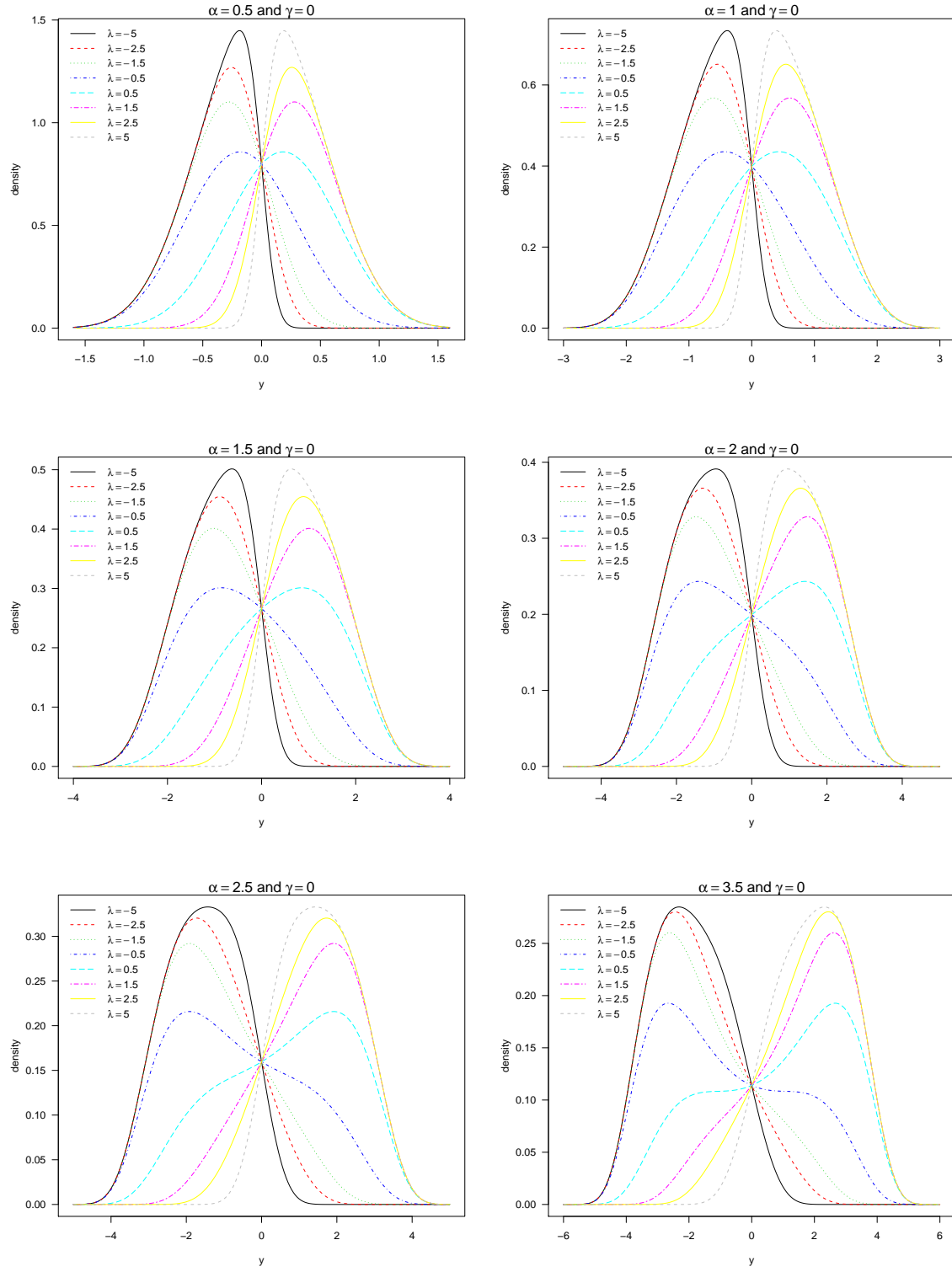


Figure 1: Plots of the density function of the SSN distribution for some parameter values.

where $\ell_i(\boldsymbol{\theta}) = -\log(2\pi)/2 + \log(\xi_{i1}) - \xi_{i2}^2/2 + \log\{\Phi(\lambda\xi_{i2})\}$,

$$\xi_{i1} = \xi_{i1}(\boldsymbol{\theta}) = \frac{2}{\alpha} \cosh\left(\frac{y_i - \mathbf{x}_i^\top \boldsymbol{\beta} + c(\alpha, \lambda)}{2}\right), \quad \xi_{i2} = \xi_{i2}(\boldsymbol{\theta}) = \frac{2}{\alpha} \sinh\left(\frac{y_i - \mathbf{x}_i^\top \boldsymbol{\beta} + c(\alpha, \lambda)}{2}\right),$$

for $i = 1, \dots, n$. The function $\ell(\boldsymbol{\theta})$ is assumed to be regular (Cox and Hinkley, 1974, Ch. 9) with respect to all $\boldsymbol{\beta}$, α and λ derivatives up to second order. Further, the $n \times p$ matrix $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_n)^\top$ is assumed to be of full rank, i.e., $\text{rank}(\mathbf{X}) = p$.

By taking the partial derivatives of the log-likelihood function with respect to $\boldsymbol{\beta}$, α and λ , we obtain the components of the score vector $\mathbf{U}_\theta = (U_\beta^\top, U_\alpha, U_\lambda)^\top$. We have $U_\beta = \mathbf{X}^\top \mathbf{s}$, where $\mathbf{s} = (s_1, \dots, s_n)^\top$ with $s_i = \{\xi_{i1}\xi_{i2} - \xi_{i2}/\xi_{i1} - \lambda\xi_{i1}\phi(\lambda\xi_{i1})/\Phi(\lambda\xi_{i2})\}/2$,

$$U_\alpha = -\frac{n}{\alpha} + \frac{1}{\alpha} \sum_{i=1}^n \xi_{i2}^2 - \frac{c_\alpha}{2} \sum_{i=1}^n \left\{ \xi_{i1}\xi_{i2} - \frac{\xi_{i2}}{\xi_{i1}} \right\} \\ + \lambda \sum_{i=1}^n \frac{\phi(\lambda\xi_{i2})}{\Phi(\lambda\xi_{i2})} \left\{ \frac{c_\alpha \xi_{i1}}{2} - \frac{\xi_{i2}}{\alpha} \right\},$$

$$U_\lambda = -\frac{c_\lambda}{2} \sum_{i=1}^n \left\{ \xi_{i1}\xi_{i2} - \frac{\xi_{i2}}{\xi_{i1}} \right\} + \frac{1}{2} \sum_{i=1}^n \frac{\phi(\lambda\xi_{i2})}{\Phi(\lambda\xi_{i2})} \{\lambda c_\lambda \xi_{i1} + 2\xi_{i2}\},$$

where

$$c_\alpha = c_\alpha(\alpha, \lambda) = 4 \int_{-\infty}^{\infty} w(4 + \alpha^2 w^2)^{-1/2} \phi(w) \Phi(\lambda w) dw, \\ c_\lambda = c_\lambda(\alpha, \lambda) = 4 \int_{-\infty}^{\infty} w \sinh^{-1}(\alpha w/2) \phi(w) \Phi(\lambda w) dw.$$

Setting these equations to zero, $\mathbf{U}_\theta = \mathbf{0}$, and solving them simultaneously yields the MLE $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\beta}}^\top, \hat{\alpha}, \hat{\lambda})^\top$ of $\boldsymbol{\theta} = (\boldsymbol{\beta}^\top, \alpha, \lambda)^\top$. These equations cannot be solved analytically and statistical software can be used to solve them numerically. For example, the BFGS method (see, Nocedal and Wright, 1999; Press et al., 2007) with analytical derivatives can be used for maximizing the log-likelihood function $\ell(\boldsymbol{\theta})$. Starting values $\boldsymbol{\beta}^{(0)}$, $\alpha^{(0)}$ and $\lambda^{(0)}$ are required. Our suggestion is to use as an initial point estimate for $\boldsymbol{\beta}$ the ordinary least squares estimate of this parameter vector, that is, $\bar{\boldsymbol{\beta}} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y}$. The initial guess for α we suggest is $\sqrt{\bar{\alpha}^2}$, where

$$\bar{\alpha}^2 = \frac{4}{n} \sum_{i=1}^n \sinh^2\left(\frac{y_i - \mathbf{x}_i^\top \bar{\boldsymbol{\beta}}}{2}\right).$$

We suggest $\lambda^{(0)} = 0$. These initial guesses worked well in the application described in Section 6.

The asymptotic inference for the parameter vector $\boldsymbol{\theta} = (\boldsymbol{\beta}^\top, \alpha, \lambda)^\top$ can be based on the normal approximation of the MLE of $\boldsymbol{\theta}$, $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\beta}}^\top, \hat{\alpha}, \hat{\lambda})^\top$. Under some regular conditions stated in Cox and Hinkley (1974, Ch. 9) that are fulfilled for the parameters in the interior of the parameter space, we have $\hat{\boldsymbol{\theta}} \stackrel{a}{\sim} \mathcal{N}_{p+2}(\boldsymbol{\theta}, \boldsymbol{\Sigma}_\theta)$, for n large, where $\stackrel{a}{\sim}$ means approximately distributed and $\boldsymbol{\Sigma}_\theta$ is the asymptotic variance-covariance matrix for $\hat{\boldsymbol{\theta}}$. The asymptotic behavior remains valid if $\boldsymbol{\Sigma}_\theta$ is

approximated by $-\ddot{\mathbf{L}}_{\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}}^{-1}$, where $-\ddot{\mathbf{L}}_{\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}}$ is the $(p+2) \times (p+2)$ observed information matrix evaluated at $\hat{\boldsymbol{\theta}}$, obtained from

$$\ddot{\mathbf{L}}_{\boldsymbol{\theta}\boldsymbol{\theta}} = \begin{bmatrix} \ddot{\mathbf{L}}_{\beta\beta} & \ddot{\mathbf{L}}_{\beta\alpha} & \ddot{\mathbf{L}}_{\beta\lambda} \\ \ddot{\mathbf{L}}_{\alpha\beta} & \ddot{\mathbf{L}}_{\alpha\alpha} & \ddot{\mathbf{L}}_{\alpha\lambda} \\ \ddot{\mathbf{L}}_{\lambda\beta} & \ddot{\mathbf{L}}_{\lambda\alpha} & \ddot{\mathbf{L}}_{\lambda\lambda} \end{bmatrix} = \begin{bmatrix} -\mathbf{X}^\top \mathbf{V} \mathbf{X} & -\mathbf{X}^\top \mathbf{h} & -\mathbf{X}^\top \mathbf{b} \\ -\mathbf{h}^\top \mathbf{X} & \text{tr}(\mathbf{K}_1) & \text{tr}(\mathbf{K}_2) \\ -\mathbf{b}^\top \mathbf{X} & \text{tr}(\mathbf{K}_2) & \text{tr}(\mathbf{K}_3) \end{bmatrix},$$

where

$$\mathbf{V} = \text{diag}\{v_1, \dots, v_n\}, \quad \mathbf{K}_1 = \text{diag}\{k_{i1}, \dots, k_{n1}\}, \quad \mathbf{K}_2 = \text{diag}\{k_{i2}, \dots, k_{n2}\}, \\ \mathbf{K}_3 = \text{diag}\{k_{i3}, \dots, k_{n3}\}, \quad \mathbf{h} = (h_1, \dots, h_n)^\top, \quad \mathbf{b} = (b_1, \dots, b_n)^\top.$$

All the quantities necessary to obtain the observed information matrix are given in the Appendix.

3 Local influence

The local influence method is recommended when the concern is related to investigate the model sensibility under some minor perturbations in the model (or data). Let $\boldsymbol{\omega} \in \boldsymbol{\Omega}$ be a k -dimensional vector of perturbations, where $\boldsymbol{\Omega} \subset \mathbb{R}^k$ is an open set. The perturbed log-likelihood function is denoted by $\ell(\boldsymbol{\theta}|\boldsymbol{\omega})$. The vector of no perturbation is $\boldsymbol{\omega}_0 \in \boldsymbol{\Omega}$, such that $\ell(\boldsymbol{\theta}|\boldsymbol{\omega}_0) = \ell(\boldsymbol{\theta})$. The influence of minor perturbations on the maximum likelihood estimate $\hat{\boldsymbol{\theta}}$ can be assessed by using the log-likelihood displacement $LD_{\boldsymbol{\omega}} = 2\{\ell(\hat{\boldsymbol{\theta}}) - \ell(\hat{\boldsymbol{\theta}}_{\boldsymbol{\omega}})\}$, where $\hat{\boldsymbol{\theta}}_{\boldsymbol{\omega}}$ denotes the maximum likelihood estimate under $\ell(\boldsymbol{\theta}|\boldsymbol{\omega})$.

The Cook's idea for assessing local influence is essentially to analyse the local behavior of $LD_{\boldsymbol{\omega}}$ around $\boldsymbol{\omega}_0$ by evaluating the curvature of the plot of $LD_{\boldsymbol{\omega}_0 + a\mathbf{d}}$ against a , where $a \in \mathbb{R}$ and \mathbf{d} is a unit norm direction. One of the measures of particular interest is the direction \mathbf{d}_{\max} corresponding to the largest curvature $C_{\mathbf{d}_{\max}}$. The index plot of \mathbf{d}_{\max} may evidence those observations that have considerable influence on $LD_{\boldsymbol{\omega}}$ under minor perturbations. Also, plots of \mathbf{d}_{\max} against covariate values may be helpful for identifying atypical patterns. Cook (1986) shows that the normal curvature at the direction \mathbf{d} is given by

$$C_{\mathbf{d}}(\boldsymbol{\theta}) = 2|\mathbf{d}^\top \boldsymbol{\Delta}^\top \ddot{\mathbf{L}}_{\boldsymbol{\theta}\boldsymbol{\theta}}^{-1} \boldsymbol{\Delta} \mathbf{d}|,$$

where $\boldsymbol{\Delta} = \partial^2 \ell(\boldsymbol{\theta}|\boldsymbol{\omega}) / \partial \boldsymbol{\theta} \partial \boldsymbol{\omega}^\top$ and $-\ddot{\mathbf{L}}_{\boldsymbol{\theta}\boldsymbol{\theta}}$ is the observed information matrix, both $\boldsymbol{\Delta}$ and $\ddot{\mathbf{L}}_{\boldsymbol{\theta}\boldsymbol{\theta}}$ are evaluated at $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{\omega}_0$. Hence, $C_{\mathbf{d}_{\max}}/2$ is the largest eigenvalue of $\mathbf{B} = -\boldsymbol{\Delta}^\top \ddot{\mathbf{L}}_{\boldsymbol{\theta}\boldsymbol{\theta}}^{-1} \boldsymbol{\Delta}$ and \mathbf{d}_{\max} is the corresponding unit norm eigenvector. The index plot of \mathbf{d}_{\max} for the matrix \mathbf{B} may show how to perturb the model (or data) to obtain large changes in the estimate of $\boldsymbol{\theta}$.

Assume that the parameter vector $\boldsymbol{\theta}$ is partitioned as $\boldsymbol{\theta} = (\boldsymbol{\theta}_1^\top, \boldsymbol{\theta}_2^\top)^\top$. The dimensions of $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ are p_1 and $p - p_1$, respectively. Let

$$\ddot{\mathbf{L}}_{\boldsymbol{\theta}\boldsymbol{\theta}} = \begin{bmatrix} \ddot{\mathbf{L}}_{\boldsymbol{\theta}_1\boldsymbol{\theta}_1} & \ddot{\mathbf{L}}_{\boldsymbol{\theta}_1\boldsymbol{\theta}_2} \\ \ddot{\mathbf{L}}_{\boldsymbol{\theta}_1\boldsymbol{\theta}_2}^\top & \ddot{\mathbf{L}}_{\boldsymbol{\theta}_2\boldsymbol{\theta}_2} \end{bmatrix},$$

where $\ddot{L}_{\theta_1\theta_1} = \partial^2\ell(\boldsymbol{\theta})/\partial\theta_1\partial\theta_1^\top$, $\ddot{L}_{\theta_1\theta_2} = \partial^2\ell(\boldsymbol{\theta})/\partial\theta_1\partial\theta_2^\top$ and $\ddot{L}_{\theta_2\theta_2} = \partial^2\ell(\boldsymbol{\theta})/\partial\theta_2\partial\theta_2^\top$. If the interest lies on θ_1 , the normal curvature in the direction of the vector \mathbf{d} is $C_{\mathbf{d};\theta_1}(\boldsymbol{\theta}) = 2|\mathbf{d}^\top \boldsymbol{\Delta}^\top (\ddot{L}_{\theta\theta}^{-1} - \ddot{L}_{22}) \boldsymbol{\Delta} \mathbf{d}|$, where

$$\ddot{L}_{22} = \begin{bmatrix} 0 & 0 \\ 0 & \ddot{L}_{\theta_2\theta_2}^{-1} \end{bmatrix}$$

and $\mathbf{d}_{\max;\theta_1}$ here is the eigenvector corresponding to the largest eigenvalue of $\mathbf{B}_1 = -\boldsymbol{\Delta}^\top (\ddot{L}_{\theta\theta}^{-1} - \ddot{L}_{22}) \boldsymbol{\Delta}$ (Cook, 1986). The index plot of the $\mathbf{d}_{\max;\theta_1}$ may reveal those influential elements on $\hat{\theta}_1$.

4 Curvature calculations

Next, we derive for three perturbation schemes the matrix

$$\boldsymbol{\Delta} = \left. \frac{\partial^2\ell(\boldsymbol{\theta}|\boldsymbol{\omega})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\omega}^\top} \right|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}, \boldsymbol{\omega}=\boldsymbol{\omega}_0} = \begin{bmatrix} \boldsymbol{\Delta}_\beta \\ \boldsymbol{\Delta}_\alpha \\ \boldsymbol{\Delta}_\lambda \end{bmatrix},$$

considering the model defined in (2) and its log-likelihood function given by (3). The quantities distinguished by the addition of “ \sim ” are evaluated at $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\beta}}^\top, \hat{\alpha}, \hat{\lambda})^\top$.

4.1 Case-weights perturbation

The perturbation of cases is done by defining some weights for each observation in the log-likelihood function as follows:

$$\ell(\boldsymbol{\theta}|\boldsymbol{\omega}) = \sum_{i=1}^n \omega_i \ell_i(\boldsymbol{\theta}),$$

where $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)^\top$ is the total vector of weights and $\boldsymbol{\omega}_0 = (1, \dots, 1)^\top$ is the vector of no perturbations. After some algebra, we have

$$\boldsymbol{\Delta}_\beta = \mathbf{X}^\top \hat{\mathbf{S}}, \quad \boldsymbol{\Delta}_\alpha = (\hat{a}_1, \dots, \hat{a}_n), \quad \boldsymbol{\Delta}_\lambda = (\hat{c}_1, \dots, \hat{c}_n),$$

where $\mathbf{S} = \text{diag}\{s_1, \dots, s_n\}$,

$$a_i = -\frac{1}{\alpha} + \frac{\xi_{i2}^2}{\alpha} - \frac{c_\alpha}{2} \left\{ \xi_{i1}\xi_{i2} - \frac{\xi_{i2}}{\xi_{i1}} \right\} + \frac{\lambda\phi(\lambda\xi_{i2})}{\Phi(\lambda\xi_{i2})} \left\{ -\frac{\xi_{i2}}{\alpha} + \frac{c_\alpha\xi_{i1}}{2} \right\},$$

$$c_i = -\frac{c_\alpha}{2} \left\{ \xi_{i1}\xi_{i2} - \frac{\xi_{i2}}{\xi_{i1}} \right\} + \frac{\phi(\lambda\xi_{i2})}{2\Phi(\lambda\xi_{i2})} (2\xi_{i2} + \lambda c_\lambda \xi_{i1}),$$

for $i = 1, \dots, n$.

4.2 Response perturbation

We shall consider here that each y_i is perturbed as $y_{iw} = y_i + \omega_i s_y$, where s_y is a scale factor that may be estimated by the standard deviation of \mathbf{y} . In this case, the perturbed log-likelihood function is given by

$$\ell(\boldsymbol{\theta}|\boldsymbol{\omega}) = -\frac{n}{2} \log(8\pi) + \sum_{i=1}^n \log(\xi_{i1w_1}) - \frac{1}{2} \sum_{i=1}^n \xi_{i2w_1}^2,$$

where $\xi_{i1w_1} = \xi_{i1w_1}(\boldsymbol{\theta}) = 2\alpha^{-1} \cosh([y_{iw} - \mathbf{x}_i^\top \boldsymbol{\beta} + c(\alpha, \lambda)]/2)$, $\xi_{i2w_1} = \xi_{i2w_1}(\boldsymbol{\theta}) = 2\alpha^{-1} \sinh([y_{iw} - \mathbf{x}_i^\top \boldsymbol{\beta} + c(\alpha, \lambda)]/2)$ and $\boldsymbol{\omega}_0 = (0, \dots, 0)^\top$ is the vector of no perturbations. Here,

$$\Delta_{\boldsymbol{\beta}} = s_y \mathbf{X}^\top \widehat{\mathbf{V}}, \quad \Delta_{\alpha} = s_y \widehat{\mathbf{h}}^\top, \quad \Delta_{\lambda} = s_y \widehat{\mathbf{b}}^\top.$$

4.3 Explanatory variable perturbation

Consider now an additive perturbation on a particular continuous explanatory variable, namely x_j , by making $x_{ijw} = x_{ij} + \omega_i s_x$, where s_x is a scale factor that may be estimated by the standard deviation of x_j . This perturbation scheme leads to the following expression for the log-likelihood function:

$$\ell(\boldsymbol{\theta}|\boldsymbol{\omega}) = -\frac{n}{2} \log(8\pi) + \sum_{i=1}^n \log(\xi_{i1w_2}) - \frac{1}{2} \sum_{i=1}^n \xi_{i2w_2}^2,$$

where $\xi_{i1w_2} = \xi_{i1w_2}(\boldsymbol{\theta}) = 2\alpha^{-1} \cosh([y_i - \mathbf{x}_{iw}^\top \boldsymbol{\beta} + c(\alpha, \lambda)]/2)$, $\xi_{i2w_2} = \xi_{i2w_2}(\boldsymbol{\theta}) = 2\alpha^{-1} \sinh([y_i - \mathbf{x}_{iw}^\top \boldsymbol{\beta} + c(\alpha, \lambda)]/2)$, with $\mathbf{x}_{iw} = (x_{i1}, \dots, x_{ijw}, \dots, x_{ip})^\top$. Here, $\boldsymbol{\omega}_0 = (0, \dots, 0)^\top$ is the vector of no perturbations. Under this perturbation scheme, we have

$$\Delta_{\boldsymbol{\beta}} = -s_x \widehat{\boldsymbol{\beta}}_j \mathbf{X}^\top \widehat{\mathbf{V}} + s_x \mathbf{c}_j \widehat{\mathbf{s}}^\top, \quad \Delta_{\alpha} = -s_x \widehat{\boldsymbol{\beta}}_j \widehat{\mathbf{h}}^\top, \quad \Delta_{\lambda} = -s_x \widehat{\boldsymbol{\beta}}_j \widehat{\mathbf{b}}^\top,$$

where \mathbf{c}_j denotes a $p \times 1$ vector with 1 at the j th position and zero elsewhere and $\widehat{\boldsymbol{\beta}}_j$ denotes the j th element of $\widehat{\boldsymbol{\beta}}$, for $j = 1, \dots, p$.

5 Generalized leverage

In what follows we shall use the generalized leverage proposed by Wei et al. (1998), which is defined as $GL(\tilde{\boldsymbol{\theta}}) = \partial \tilde{\mathbf{y}} / \partial \mathbf{y}^\top$, where $\boldsymbol{\theta}$ is an s -vector such that $E(\mathbf{y}) = \boldsymbol{\mu}(\boldsymbol{\theta})$ and $\tilde{\boldsymbol{\theta}}$ is an estimator of $\boldsymbol{\theta}$, with $\tilde{\mathbf{y}} = \boldsymbol{\mu}(\tilde{\boldsymbol{\theta}})$. Here, the (i, l) element of $GL(\tilde{\boldsymbol{\theta}})$, i.e. the generalized leverage of the estimator $\tilde{\boldsymbol{\theta}}$ at (i, l) , is the instantaneous rate of change in i th predicted value with respect to the l th response value. As noted by the authors, the generalized leverage is invariant under reparameterization and observations with large GL_{ij} are leverage points. Wei et al. (1998) have shown that the generalized leverage is obtained by evaluating

$$GL(\boldsymbol{\theta}) = \mathbf{D}_{\boldsymbol{\theta}} (-\ddot{\mathbf{L}}_{\boldsymbol{\theta}\boldsymbol{\theta}})^{-1} \ddot{\mathbf{L}}_{\boldsymbol{\theta}\mathbf{y}},$$

at $\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}$, where $\mathbf{D}_{\boldsymbol{\theta}} = \partial \boldsymbol{\mu} / \partial \boldsymbol{\theta}^\top$ and $\ddot{\mathbf{L}}_{\boldsymbol{\theta}\mathbf{y}} = \partial^2 \ell(\boldsymbol{\theta}) / \partial \boldsymbol{\theta} \partial \mathbf{y}^\top$.

After some algebra, we have that

$$D_{\theta} = \begin{bmatrix} X & 0 & 0 \end{bmatrix} \quad \text{and} \quad \ddot{L}_{\theta y} = - \begin{bmatrix} X^{\top} V \\ h^{\top} \\ b^{\top} \end{bmatrix}.$$

Thus, from these quantities, we can obtain the generalized leverage.

6 Application

In this section we shall illustrate the usefulness of the proposed regression model. The fatigue processes are by excellence ideally modeled by the Birnbaum–Saunders distribution due to its genesis. We consider the data set given in McCool (1980) and reported in Chan et al. (2008). These data consist of times to failure (T) in rolling contact fatigue of ten hardened steel specimens tested at each of four values of four contact stress (x). The data were obtained using a 4-ball rolling contact test rig at the Princeton Laboratories of Mobil Research and Development Co. Similarly to Chan et al. (2008), we consider the following regression model:

$$y_i = \beta_1 + \beta_2 \log(x_i) + \varepsilon_i, \quad i = 1, \dots, 40,$$

where $y_i = \log(T_i)$ and $\varepsilon_i \sim \text{SSN}(\alpha, -c(\alpha, \lambda), 2, \lambda)$, for $i = 1, \dots, 40$. All the computations were done using the Ox matrix programming language (Doornik, 2006). Ox is freely distributed for academic purposes and available at <http://www.doornik.com>.

Table 1: Maximum likelihood estimates.

Parameter	log-BS		skewed log-BS	
	Estimate	SE	Estimate	SE
β_1	0.0978	0.1707	0.1657	0.1759
β_2	-14.1164	1.5714	-13.8710	1.5887
α	1.2791	0.1438	2.0119	0.3487
λ	—	—	1.6423	0.5679
Log-likelihood	-61.62		-58.68	
AIC	129.24		125.36	
BIC	134.31		132.12	
HQIC	131.07		127.80	

Table 1 lists the MLEs of the model parameters, asymptotic standard errors (SE), the values of the log-likelihood functions and the statistics AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion) and HQIC (Hannan-Quinn Information Criterion) for the skewed log-BS and log-BS regression models. The SE of the estimates for the skewed log-BS model were obtained using

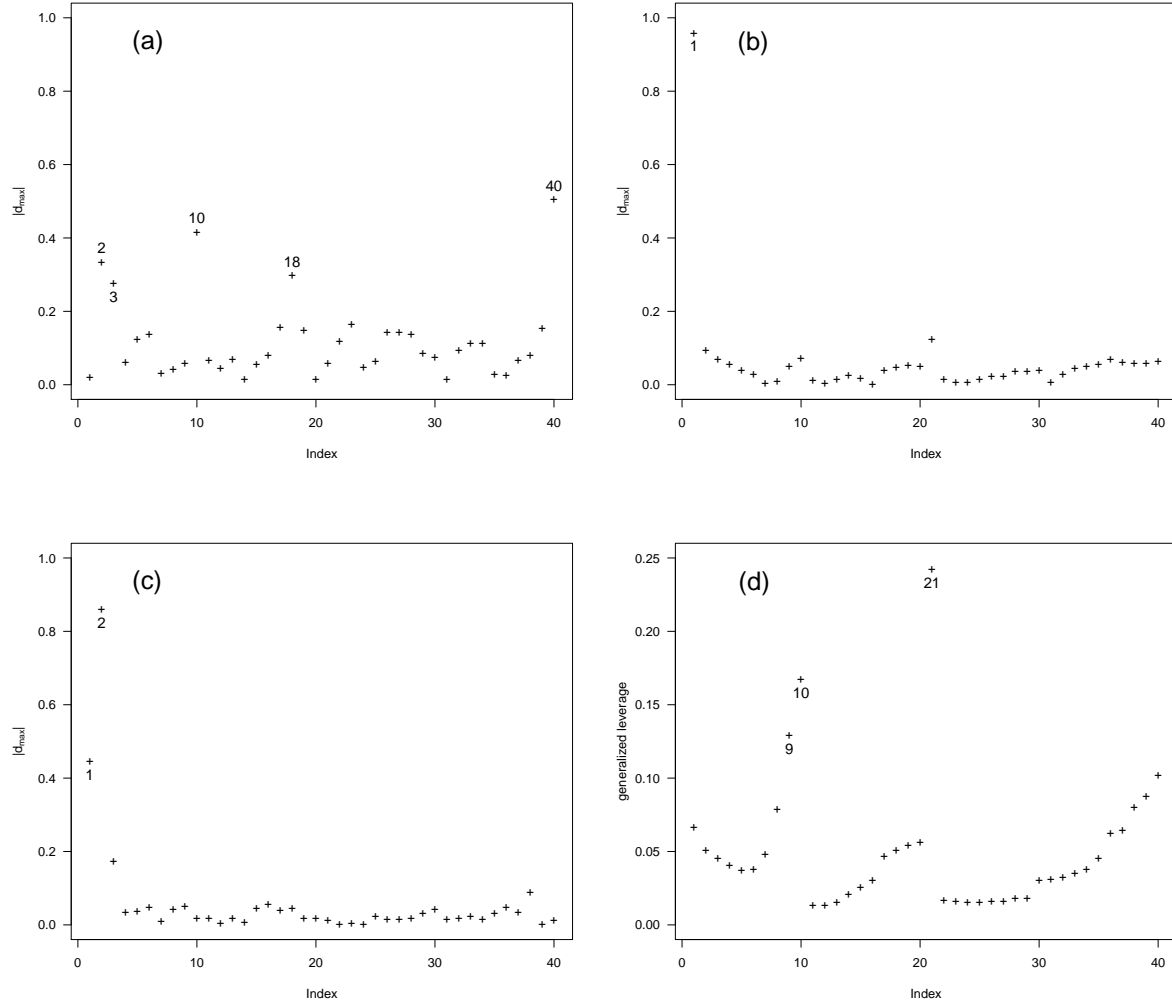


Figure 2: Index plots of $|d_{\max}|$ for $\hat{\theta}$ under case weighting (a), response (b) and covariate (c) perturbations, and generalized leverage (d).

the observed information matrix given in Section 2, while the SE of the estimates for the log-BS model were obtained using the observed information matrix given, for example, in Galea et al. (2004). The estimatives of β_1 and β_2 differ slightly between the two models. The skewed log-BS model yields the highest value of the log-likelihood function and smallest values of the AIC, BIC and HQIC statistics. From the values of these statistics, the skewed log-BS model outperforms the BS model and should be preferred. The likelihood ratio (LR) statistic to the null hypothesis $\lambda = 0$ is in accordance with the information criteria (LR = 5.88 and the associated critical level of the χ_1^2 at 5% is 3.84).

In what follows, we shall apply the generalized leverage and local influence methods developed in the previous sections for the purpose of identifying influential observations in the skewed log-BS regression model fitted to the data set. Figure 2 gives the $|d_{\max}|$ corresponding to $\hat{\theta}$ for different perturbation schemes and the generalized leverage. An inspection of Figure 2 reveals that based on

case-weight perturbation (Figure 2(a)), we observed that the cases #2, #3, #10, #18 and #40 have more pronounced influence than the other observations. The case #1 appears with outstanding influence based on response perturbation (Figures 2(b)). From the Figure 2(c) (covariate perturbation), the case #1 and #2 have more pronounced influence than the other observations. Figure 2(d) reveals that the cases #9, #10 and #21 have influence on their own-fitted values.

Based on Figure 2, we eliminated those most influential observations and refitted the skewed log-BS regression model. In Table 2 we have the relative changes of each parameter estimate, defined by $RC = |(\hat{\theta}_j - \hat{\theta}_{j(i)})/\hat{\theta}_j|$, and the corresponding SE, where $\hat{\theta}_{j(i)}$ denotes the maximum likelihood estimate of θ_j , after removing the i th observation. As can be seen, except for the case #21 corresponding to the parameter λ , the relative changes for the maximum likelihood estimates of β_2 , α and λ are very little pronounced. Also, the significance of these parameters are not modified in all cases considered. Case #21 represents the smallest value of the time to failure. Further, β_1 becomes not significant in all cases considered similar to the skewed log-BS regression model fitted considering all observations (Table 1).

Table 2: Relative changes dropping the cases indicated, and the corresponding asymptotic standard errors.

	β_1		β_2		α		λ	
Dropping	RC	SE	RC	SE	RC	SE	RC	SE
#1	0.201	0.180	0.029	1.614	0.026	0.341	0.035	0.543
#2	0.161	0.180	0.024	1.618	0.013	0.345	0.030	0.544
#3	0.145	0.180	0.022	1.619	0.009	0.347	0.030	0.544
#9	0.304	0.181	0.033	1.665	0.027	0.369	0.068	0.609
#10	0.543	0.180	0.051	1.675	0.005	0.358	0.044	0.588
#18	0.336	0.177	0.011	1.558	0.015	0.347	0.000	0.569
#21	0.549	0.132	0.173	1.125	0.913	0.150	3.312	0.406
#40	0.121	0.176	0.027	1.626	0.014	0.346	0.019	0.565

7 Concluding remarks

In this paper we have introduced a log-Birnbaum–Saunders regression model with asymmetric errors, extending the usual log-BS regression model. The random errors of the regression model follow a skewed sinh-normal distribution, recently derived by Leiva et al. (2010). The estimation of the model parameters is approached by the method of maximum likelihood and the observed information matrix is derived. We also consider diagnostic techniques that can be employed to identify influential observations. Appropriate matrices for assessing local influence on the parameter estimates under different perturbation schemes are obtained. The expressions derived are simple, compact and can be

easily implemented into any mathematical or statistical/econometric programming environment with numerical linear algebra facilities, such as R (R Development Core Team, 2009) and Ox (Doornik, 2006), among others, i.e. our formulas related with this class of regression model are manageable, and with the use of modern computer resources, may turn into adequate tools comprising the arsenal of applied statisticians. Finally, an application to a real data set is presented to illustrate the usefulness of the proposed model.

As future research, it should be noticed that some generalizations of the proposed model could be done. For example, a skewed log-BS regression model that allows us consider censored samples could be introduced. Following Xi and Wei (2007), one could introduce a skewed log-BS regression model in which the parameter α is considered different for each observation, i.e. to propose an heteroscedastic skewed log-BS regression model. Also, a skewed log-BS nonlinear regression model could be proposed, and so forth.

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Appendix

After extensive algebraic manipulations, the quantities necessary to obtain the observed information matrix for the parameter vector $\boldsymbol{\theta} = (\boldsymbol{\beta}^\top, \alpha, \lambda)^\top$ presented in the Section 2 are given by

$$v_i = v_i(\boldsymbol{\theta}) = \frac{1}{4} \left\{ 2\xi_{i2}^2 + \frac{4}{\alpha^2} - 1 + \frac{\xi_{i2}^2}{\xi_{i1}^2} - \frac{\lambda\xi_{i2}\phi(\lambda\xi_{i2})}{\Phi(\lambda\xi_{i2})} + \frac{\lambda^3\xi_{i1}^2\xi_{i2}\phi(\lambda\xi_{i2})}{\Phi(\lambda\xi_{i2})} + \frac{\lambda^2\xi_{i1}^2\phi(\lambda\xi_{i2})^2}{\Phi(\lambda\xi_{i2})^2} \right\},$$

$$\begin{aligned} h_i = h_i(\boldsymbol{\theta}) &= \frac{\xi_{i1}\xi_{i2}}{\alpha} - \frac{c_\alpha}{4} \left\{ 2\xi_{i2}^2 + \frac{4}{\alpha^2} - 1 + \frac{\xi_{i2}^2}{\xi_{i1}^2} \right\} \\ &+ \frac{\lambda c_\alpha}{4} \left\{ \frac{\xi_{i2}\phi(\lambda\xi_{i2})}{\Phi(\lambda\xi_{i2})} - \frac{\lambda^2\xi_{i1}^2\xi_{i2}\phi(\lambda\xi_{i2})}{\Phi(\lambda\xi_{i2})} - \frac{\lambda\xi_{i1}^2\phi(\lambda\xi_{i2})^2}{\Phi(\lambda\xi_{i2})^2} \right\} \\ &- \frac{\lambda}{2\alpha} \left\{ \frac{\xi_{i1}\phi(\lambda\xi_{i2})}{\Phi(\lambda\xi_{i2})} - \frac{\lambda^2\xi_{i1}\xi_{i2}^2\phi(\lambda\xi_{i2})}{\Phi(\lambda\xi_{i2})} - \frac{\lambda\xi_{i1}\xi_{i2}\phi(\lambda\xi_{i2})^2}{\Phi(\lambda\xi_{i2})^2} \right\}, \end{aligned}$$

$$\begin{aligned} b_i = b_i(\boldsymbol{\theta}) &= -\frac{c_\lambda}{4} \left\{ 2\xi_{i2}^2 + \frac{4}{\alpha^2} - 1 + \frac{\xi_{i2}^2}{\xi_{i1}^2} \right\} \\ &+ \frac{\lambda c_\lambda}{4} \left\{ \frac{\xi_{i2}\phi(\lambda\xi_{i2})}{\Phi(\lambda\xi_{i2})} - \frac{\lambda^2\xi_{i1}^2\xi_{i2}\phi(\lambda\xi_{i2})}{\Phi(\lambda\xi_{i2})} - \frac{\lambda\xi_{i1}^2\phi(\lambda\xi_{i2})^2}{\Phi(\lambda\xi_{i2})^2} \right\} \\ &+ \frac{1}{2} \left\{ \frac{\xi_{i1}\phi(\lambda\xi_{i2})}{\Phi(\lambda\xi_{i2})} - \frac{\lambda^2\xi_{i1}\xi_{i2}^2\phi(\lambda\xi_{i2})}{\Phi(\lambda\xi_{i2})} - \frac{\lambda\xi_{i1}\xi_{i2}\phi(\lambda\xi_{i2})^2}{\Phi(\lambda\xi_{i2})^2} \right\}, \end{aligned}$$

$$\begin{aligned}
k_{i1} = k_{i1}(\boldsymbol{\theta}) &= \frac{1}{\alpha^2} - \frac{3\xi_{i2}^2}{\alpha^2} - \frac{c'_\alpha}{2} \left\{ \xi_{i1}\xi_{i2} - \frac{\xi_{i2}}{\xi_{i1}} \right\} + \frac{\lambda c'_\alpha \xi_{i1} \phi(\lambda \xi_{i2})}{2\Phi(\lambda \xi_{i2})} + \frac{c_\alpha \xi_{i1} \xi_{i2}}{\alpha} \\
&\quad - \frac{c'_\alpha}{2} \left\{ -\frac{2\xi_{i1}\xi_{i2}}{\alpha} + \frac{c_\alpha}{2}(\xi_{i1}^2 + \xi_{i2}^2) - \frac{2c_\alpha}{\alpha^2 \xi_{i1}^2} \right\} - \frac{\lambda \phi(\lambda \xi_{i2})}{\alpha \Phi(\lambda \xi_{i2})} \left\{ -\frac{\xi_{i2}}{\alpha} + \frac{c_\alpha \xi_{i1}}{2} \right\} \\
&\quad + \frac{\lambda c_\alpha \phi(\lambda \xi_{i2})}{2\Phi(\lambda \xi_{i2})} \left\{ -\frac{\xi_{i1}}{\alpha} + \frac{c_\alpha \xi_{i2}}{2} \right\} + \frac{\lambda \xi_{i2} \phi(\lambda \xi_{i2})}{\alpha^2 \Phi(\lambda \xi_{i2})} \\
&\quad - \frac{\lambda^2 c_\alpha \xi_{i1} \phi(\lambda \xi_{i2})}{2\Phi(\lambda \xi_{i2})} \left\{ -\frac{\xi_{i2}}{\alpha} + \frac{c_\alpha \xi_{i1}}{2} \right\} \left\{ \lambda \xi_{i2} + \frac{\phi(\lambda \xi_{i2})}{\Phi(\lambda \xi_{i2})} \right\} \\
&\quad + \frac{\lambda^2 \xi_{i2} \phi(\lambda \xi_{i2})}{\alpha \Phi(\lambda \xi_{i2})} \left\{ -\frac{\xi_{i2}}{\alpha} + \frac{c_\alpha \xi_{i1}}{2} \right\} \left\{ \lambda \xi_{i2} + \frac{\phi(\lambda \xi_{i2})}{\Phi(\lambda \xi_{i2})} \right\},
\end{aligned}$$

$$\begin{aligned}
k_{i2} = k_{i2}(\boldsymbol{\theta}) &= -\frac{c_{\alpha\lambda}}{2} \left\{ \xi_{i1}\xi_{i2} - \frac{\xi_{i2}}{\xi_{i1}} \right\} + \frac{\lambda c_{\alpha\lambda} \xi_{i1} \phi(\lambda \xi_{i2})}{2\Phi(\lambda \xi_{i2})} \\
&\quad - \frac{c_\lambda}{2} \left\{ -\frac{2\xi_{i1}\xi_{i2}}{\alpha} + \frac{c_\alpha}{2}(\xi_{i1}^2 + \xi_{i2}^2) - \frac{2c_\alpha}{\alpha^2 \xi_{i1}^2} \right\} \\
&\quad + \frac{\lambda c_\lambda \phi(\lambda \xi_{i2})}{2\Phi(\lambda \xi_{i2})} \left\{ -\frac{\xi_{i1}}{\alpha} + \frac{c_\alpha \xi_{i2}}{2} \right\} + \frac{\phi(\lambda \xi_{i2})}{\Phi(\lambda \xi_{i2})} \left\{ -\frac{\xi_{i2}}{\alpha} + \frac{c_\alpha \xi_{i1}}{2} \right\} \\
&\quad - \frac{\lambda^2 c_\lambda \xi_{i1} \phi(\lambda \xi_{i2})}{2\Phi(\lambda \xi_{i2})} \left\{ -\frac{\xi_{i2}}{\alpha} + \frac{c_\alpha \xi_{i1}}{2} \right\} \left\{ \lambda \xi_{i2} + \frac{\phi(\lambda \xi_{i2})}{\Phi(\lambda \xi_{i2})} \right\} \\
&\quad - \frac{\lambda \xi_{i2} \phi(\lambda \xi_{i2})}{\alpha \Phi(\lambda \xi_{i2})} \left\{ -\frac{\xi_{i2}}{\alpha} + \frac{c_\alpha \xi_{i1}}{2} \right\} \left\{ \lambda \xi_{i2} + \frac{\phi(\lambda \xi_{i2})}{\Phi(\lambda \xi_{i2})} \right\},
\end{aligned}$$

$$\begin{aligned}
k_{i3} = k_{i3}(\boldsymbol{\theta}) &= -\frac{c'_\lambda}{2} \left\{ \xi_{i1}\xi_{i2} - \frac{\xi_{i2}}{\xi_{i1}} \right\} + \frac{c_\lambda \xi_{i1} \phi(\lambda \xi_{i2})}{\Phi(\lambda \xi_{i2})} + \frac{\lambda c'_\lambda \xi_{i1} \phi(\lambda \xi_{i2})}{2\Phi(\lambda \xi_{i2})} \\
&\quad - \frac{c_\lambda^2}{4} \left\{ 2\xi_{i2}^2 + \frac{4}{\alpha^2} - 1 + \frac{\xi_{i2}^2}{\xi_{i1}^2} \right\} + \frac{\lambda c_\lambda^2 \xi_{i2} \phi(\lambda \xi_{i2})}{4\Phi(\lambda \xi_{i2})} \\
&\quad - \frac{\lambda^2 c_\lambda \xi_{i1} \xi_{i2} \phi(\lambda \xi_{i2})}{2\Phi(\lambda \xi_{i2})} \left\{ \xi_{i2} + \frac{\lambda c_\lambda \xi_{i1}}{2} \right\} - \frac{\lambda c_\lambda \xi_{i1} \phi(\lambda \xi_{i2})^2}{2\Phi(\lambda \xi_{i2})^2} \left\{ \xi_{i2} + \frac{\lambda c_\lambda \xi_{i1}}{2} \right\} \\
&\quad - \frac{\lambda \xi_{i2}^2 \phi(\lambda \xi_{i2})}{2\Phi(\lambda \xi_{i2})} \left\{ \xi_{i2} + \frac{\lambda c_\lambda \xi_{i1}}{2} \right\} - \frac{\xi_{i2} \phi(\lambda \xi_{i2})^2}{2\Phi(\lambda \xi_{i2})^2} \left\{ \xi_{i2} + \frac{\lambda c_\lambda \xi_{i1}}{2} \right\},
\end{aligned}$$

for $i = 1, \dots, n$. Also,

$$c'_\alpha = c'_\alpha(\alpha, \lambda) = -4\alpha \int_{-\infty}^{\infty} w^3 (4 + \alpha^2 w^2)^{-3/2} \phi(w) \Phi(\lambda w) dw,$$

$$c'_\lambda = c'_\lambda(\alpha, \lambda) = -4\lambda \int_{-\infty}^{\infty} w^3 \sinh^{-1}(\alpha w/2) \phi(w) \phi(\lambda w) dw,$$

$$c_{\alpha\lambda} = c_{\alpha\lambda}(\alpha, \lambda) = 4 \int_{-\infty}^{\infty} w^2 (4 + \alpha^2 w^2)^{-1/2} \phi(w) \phi(\lambda w) dw.$$

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